

# ON BESOV REGULARITY OF BROWNIAN MOTIONS IN INFINITE DIMENSIONS

TUOMAS HYTÖNEN AND MARK VERAAR

**ABSTRACT.** We extend to the vector-valued situation some earlier work of Ciesielski and Roynette on the Besov regularity of the paths of the classical Brownian motion. We also consider a Brownian motion as a Besov space valued random variable. It turns out that a Brownian motion, in this interpretation, is a Gaussian random variable with some pathological properties. We prove estimates for the first moment of the Besov norm of a Brownian motion. To obtain such results we estimate expressions of the form  $\mathbb{E} \sup_{n \geq 1} \|\xi_n\|$ , where the  $\xi_n$  are independent centered Gaussian random variables with values in a Banach space. Using isoperimetric inequalities we obtain two-sided inequalities in terms of the first moments and the weak variances of  $\xi_n$ .

## 1. INTRODUCTION

Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space. Let  $W : [0, 1] \times \Omega \rightarrow \mathbb{R}$  be a standard Brownian motion. Since  $W$  has continuous paths, it is easy to check that  $W : \Omega \rightarrow C([0, 1])$  is a  $C([0, 1])$ -valued Gaussian random variable. Moreover, since  $W$  is  $\alpha$ -Hölder continuous for all  $\alpha \in (0, \frac{1}{2})$  one can also show that for all  $0 < \alpha < 1/2$ ,  $W : \Omega \rightarrow C^\alpha([0, 1])$  is a Gaussian random variable. In this way one obtains results like

$$\mathbb{E} \exp(\varepsilon \|W\|_{C^\alpha([0, 1])}^2) < \infty$$

for some  $\varepsilon > 0$ .

In [2, 3] Ciesielski has improved the Hölder continuity results of Brownian motion using Besov spaces. He has proved that almost all paths of  $W$  are in the Besov space  $B_{p, \infty}^{1/2}(0, 1)$  for all  $p \in [1, \infty)$  or even in the Besov-Orlicz space  $B_{\Phi_2, \infty}^{1/2}(0, 1)$ , where  $\Phi_2(x) = e^{x^2} - 1$  (for the definition we refer to Section 2). In [11] Roynette has characterized the set of indices  $\alpha, p, q$  for which the paths of Brownian motion belong the Besov spaces  $B_{p, q}^\alpha(0, 1)$ .

The proofs of the above results are based on certain coordinate expansions of the Brownian motion and descriptions of the Besov norms in terms of the corresponding expansion coefficients of a function. We will give more direct proofs of these results which employ the usual modulus-of-continuity definition of the Besov norms. Our methods also carry over to the vector-valued situation.

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Let  $X$  be a real Banach space. We will write  $a \lesssim b$  if there exists a universal constant  $C > 0$  such that  $a \leq Cb$ , and  $a \approx b$  if  $a \lesssim b \lesssim a$ . If the constant  $C$  is allowed to depend on some parameter  $t$ , we write  $a \lesssim_t b$  and  $a \approx_t b$  instead. Let  $(l^\Theta, \|\cdot\|_\Theta)$  denote the Orlicz sequence space with  $\Theta(x) = x^2 e^{-\frac{1}{x^2}}$ . Let  $(\xi_n)_{n \geq 1}$  be independent centered  $X$ -valued Gaussian random variables with weak variances  $(\sigma_n)_{n \geq 1}$  and  $m = \sup_{n \geq 1} \mathbb{E}\|\xi_n\|$ . In Section 3 we will show that

$$(1.1) \quad \mathbb{E} \sup_{n \geq 1} \|\xi_n\| \approx m + \|(\sigma_n)_{n \geq 1}\|_\Theta.$$

As a consequence of the Kahane–Khinchine inequalities a similar estimate holds for  $(\mathbb{E} \sup_{n \geq 1} \|\xi_n\|^p)^{1/p}$  for all  $p \in [1, \infty)$  as well, at the cost of replacing  $\approx$  by  $\approx_p$ . The proof of (1.1) is based on isoperimetric inequalities for Gaussian random variables (cf. [9]).

In Section 4 we obtain regularity properties of  $X$ -valued Brownian motions  $W$ . In particular we show that for the paths of an  $X$ -valued Brownian motion  $W$  we have  $W \in B_{p,\infty}^{1/2}(0, 1; X)$  for all  $p \in [1, \infty)$  or even  $W \in B_{\Phi_2,\infty}^{1/2}(0, 1; X)$ . Thus we can consider the mappings  $W : \Omega \rightarrow B_{p,\infty}^{1/2}(0, 1; X)$  and  $W : \Omega \rightarrow B_{\Phi_2,\infty}^{1/2}(0, 1; X)$ . A natural question is whether  $W$  is a Gaussian random variable with values in one of these spaces. To answer this some problems have to be solved, because the Banach spaces  $B_{p,\infty}^{1/2}(0, 1)$  and  $B_{\Phi_2,\infty}^{1/2}(0, 1)$  are non-separable. It will be shown in Section 5 that  $W$  is indeed a Gaussian random variable, but it has some peculiar properties. For instance we find that there exists an  $\varepsilon > 0$  such that

$$\mathbb{P}(\|W\|_{B_{p,\infty}^{1/2}(0,1;X)} \leq \varepsilon) = \mathbb{P}(\|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)} \leq \varepsilon) = 0$$

which is rather counterintuitive for a centered Gaussian random variable. It implies in particular that  $W$  is not Radon. In the last Section 6 we apply the results from Section 3 to obtain explicit estimates for  $\mathbb{E}\|W\|_{B_{p,\infty}^{1/2}(0,1;X)}$  and  $\mathbb{E}\|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)}$ .

## 2. PRELIMINARIES

**2.1. Orlicz spaces.** We briefly recall the definition of Orlicz spaces. More details can be found in [7, 10, 14].

Let  $(S, \Sigma, \mu)$  be a  $\sigma$ -finite measure space and let  $X$  be a Banach space. Let  $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$  be an even convex function such that  $\Phi(0) = 0$  and  $\lim_{x \rightarrow \infty} \Phi(x) = \infty$ . The Orlicz space  $L^\Phi(S; X)$  is defined as the set of all strongly measurable functions  $f : S \rightarrow X$  (identifying functions which are equal  $\mu$ -a.e.) with the property that there exists a  $\delta > 0$  such that

$$M_\Phi(f/\delta) := \int_S \Phi(\|f(s)\|/\delta) d\mu(s) < \infty.$$

This space is a vector space and we define

$$\rho_\Phi(f) = \inf\{\delta > 0 : M_\Phi(f/\delta) \leq 1\}.$$

The mapping  $\rho_\Phi$  defines a norm on  $L^\Phi(S; X)$  and it turns  $L^\Phi(S; X)$  into a Banach space. It is usually referred to as the *Luxemburg norm*.

For  $f \in L^\Phi(S; X)$  we also define the *Orlicz norm*

$$\|f\|_\Phi = \inf_{\delta > 0} \left\{ \frac{1}{\delta} (1 + M_\Phi(\delta f)) \right\}.$$

Usually the Orlicz norm is defined in a different way using duality, but the above norm gives exactly the same number (cf. [10, Theorem III.13]).

The two norms are equivalent, as shown in the following:

**Lemma 2.1.** *For all  $f \in L^\Phi(S; X)$  we have*

$$\rho_\Phi(f) \leq \|f\|_\Phi \leq 2\rho_\Phi(f).$$

*Proof.* Let  $\delta > 0$  be such that  $M_\Phi(f\delta) \leq 1$ . Then

$$\frac{1}{\delta}(1 + M_\Phi(\delta f)) \leq \frac{2}{\delta}.$$

Taking the infimum over all  $\delta > 0$  such that  $M_\Phi(f\delta) \leq 1$  gives the second inequality.

For the first inequality, choose  $\alpha > \|f\|_\Phi$ . Then there exists a  $\delta > 0$  such that

$$\frac{1}{\delta}(1 + M_\Phi(\delta f)) \leq \alpha.$$

Since  $\Phi(0) = 0$  and  $\Phi$  is convex it holds that  $\Phi(x/\beta) \leq \Phi(x)/\beta$  for all  $x \in \mathbb{R}$  and  $\beta \geq 1$ . Noting that  $\alpha\delta \geq 1$  it follows that

$$M_\Phi(f/\alpha) = M_\Phi\left(\frac{\delta f}{\delta\alpha}\right) \leq \frac{M_\Phi(\delta f)}{\delta\alpha} \leq 1.$$

Since  $\rho_\Phi(f)$  is the infimum over all  $\alpha > 0$  for which the previous inequality holds, and it holds for every  $\alpha > \|f\|_\Phi$ , we conclude that  $\rho_\Phi(f) \leq \|f\|_\Phi$ .  $\square$

It is clear from the proof that the lemma holds for all functions  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  that satisfy  $\Phi(0) = 0$  and  $\Phi(x/\beta) \leq \Phi(x)/\beta$  for all  $x \in \mathbb{R}_+$  and  $\beta \geq 1$ . An interesting example of a non-convex function that satisfies the above properties is  $\Phi(x) = xe^{-1/x^2}$ .

**2.2. The Orlicz sequence space  $l^\Theta$ .** We next present a particular Orlicz space which plays an important role in our studies. The underlying measure space is now  $\mathbb{Z}_+$  with the counting measure, and we consider the function  $\Theta : \mathbb{R} \rightarrow \mathbb{R}_+$  defined by

$$(2.1) \quad \Theta(x) = x^2 \exp\left(-\frac{1}{2x^2}\right).$$

This function satisfies the assumptions in Subsection 2.1 and we can associate an Orlicz sequence space  $l^\Theta$  to it. Thus  $l^\Theta$  consists of all sequences  $a := (a_n)_{n \geq 1}$  for which

$$\rho_\Theta(a) := \inf \left\{ \delta > 0 : \sum_{n \geq 1} \frac{a_n^2}{\delta^2} \exp\left(-\frac{\delta^2}{2a_n^2}\right) \leq 1 \right\} < \infty.$$

The following example illustrates the behaviour of  $\rho_\Theta(a)$ , but also plays a role later on.

*Example 2.1.* If  $a_n = \alpha^n$ , where  $\alpha \in [1/2, 1)$ , then  $\rho_\Theta(a) \approx \sqrt{\log(1-\alpha)^{-1}}$ .

This may be compared with  $\|a\|_p \approx (1-\alpha)^{-1/p}$ , again for  $\alpha \in [1/2, 1)$ , and  $p \in [1, \infty]$ .

*Proof.* We consider the equivalent Orlicz norm  $\|a\|_{\Theta}$ . On the one hand,

$$\begin{aligned} \sum_{n \geq 1} \lambda^2 \alpha^{2n} \exp\left(-\frac{1}{2\lambda^2 \alpha^{2n}}\right) &\leq \sum_{n \geq 1} \lambda^2 \alpha^{2n} \exp\left(-\frac{1}{2\lambda^2 \alpha^2}\right) \\ &= \frac{\lambda^2 \alpha^2}{1 - \alpha^2} \exp\left(-\frac{1}{2\lambda^2 \alpha^2}\right) \\ &\leq \frac{\lambda^2}{1 - \alpha} \exp\left(-\frac{1}{2\lambda^2}\right). \end{aligned}$$

On the other hand, let  $N \in \mathbb{Z}_+$  be such that  $\alpha^{2N} \leq 1/2 < \alpha^{2(N-1)}$ . Then

$$\begin{aligned} \sum_{n \geq 1} \lambda^2 \alpha^{2n} \exp\left(-\frac{1}{2\lambda^2 \alpha^{2n}}\right) &\geq \sum_{n=1}^N \lambda^2 \alpha^{2n} \exp\left(-\frac{1}{2\lambda^2 \alpha^{2N}}\right) \\ &\geq \lambda^2 \alpha^2 \frac{1 - \alpha^{2N}}{1 - \alpha^2} \exp\left(-\frac{1}{\lambda^2 \alpha^2}\right) \\ &\geq \frac{\lambda^2}{12(1 - \alpha)} \exp\left(-\frac{4}{\lambda^2}\right). \end{aligned}$$

From these observations it follows that

$$\|a\|_{\Theta} = \inf_{\lambda > 0} \frac{1}{\lambda} (1 + M_{\Theta}(\lambda a)) \approx \inf_{\lambda > 0} \frac{1}{\lambda} \left(1 + \frac{\lambda^2}{1 - \alpha} e^{-1/2\lambda^2}\right) =: \inf_{\lambda > 0} F(\lambda).$$

The differentiable function  $F$  tends to  $\infty$  as  $\lambda \rightarrow 0$  or  $\lambda \rightarrow \infty$ , so its infimum is attained at a point where  $F'(\lambda) = 0$ . Since

$$F'(\lambda) = -\lambda^{-2} + (1 - \alpha)^{-1} e^{-1/2\lambda^2} + (1 - \alpha)^{-1} e^{-1/2\lambda^2} \lambda^{-2},$$

where the middle-term is always positive,  $F'(\lambda) = 0$  can only happen if

$$(1 - \alpha)^{-1} e^{-1/2\lambda^2} \leq 1 \quad \text{i.e.,} \quad \lambda^{-1} \geq \lambda_0^{-1} := \sqrt{2 \log(1 - \alpha)^{-1}}.$$

But  $1/\lambda$  is the first term in  $F(\lambda)$ , so we have proved that  $F(\lambda) \gtrsim \sqrt{\log(1 - \alpha)^{-1}}$  whenever  $0 < \lambda \leq \lambda_0$ , and moreover there holds  $F(\lambda_0) \approx \sqrt{\log(1 - \alpha)^{-1}}$ , which completes the proof.  $\square$

**2.3. Besov spaces.** We recall the definition of the vector-valued Besov spaces. For the real case we refer to [12] and for the vector-valued Besov space we will give the treatise from [6].

Let  $X$  be a real Banach space and let  $I = (0, 1)$ . For  $\alpha \in (0, 1)$ ,  $p, q \in [1, \infty]$  the *vector-valued Besov space*  $B_{p,q}^{\alpha}(I; X)$  is defined as the space of all functions  $f \in L^p(I; X)$  for which the seminorm (with the usual modification for  $q = \infty$ )

$$\left( \int_0^1 (t^{-\alpha} \omega_p(f, t))^q \frac{dt}{t} \right)^{1/q}$$

is finite. Here

$$\omega_p(f, t) = \sup_{|h| \leq t} \|s \mapsto f(s + h) - f(s)\|_{L^p(I(h); X)}$$

with  $I(h) = \{s \in I : s + h \in I\}$ . The sum of the  $L^p$ -norm and this seminorm turn  $B_{p,q}^{\alpha}(I; X)$  into a Banach space. By a dyadic approximation argument (see [6,

Corollary 3.b.9]) one can show that the above seminorm is equivalent to

$$\|f\|_{p,q,\alpha} := \left( \sum_{n \geq 0} (2^{n\alpha} \|s \mapsto f(s + 2^{-n}) - f(s)\|_{L^p(I(2^{-n}); X)})^q \right)^{1/q}$$

For the purposes below it will be convenient to take

$$\|f\|_{B_{p,q}^\alpha(I; X)} = \|f\|_{L^p(I; X)} + \|f\|_{p,q,\alpha}$$

as a Banach space norm on  $B_{p,q}^\alpha(I; X)$ .

For  $0 < \beta < \infty$ , we also introduce the exponential Orlicz and Orlicz–Besov (semi)norms

$$\begin{aligned} \|f\|_{\mathfrak{L}^{\Phi_\beta}(I; X)} &:= \sup_{p \geq 1} p^{-1/\beta} \|f\|_{L^p(I; X)}, \\ \|f\|_{\Phi_\beta, \infty, \alpha} &:= \sup_{n \geq 1} 2^{\alpha n} \|f - f(\cdot - 2^{-n})\|_{\mathfrak{L}^{\Phi_\beta}(I(2^{-n}); X)} = \sup_{p \geq 1} p^{-1/\beta} \|f\|_{p, \infty, \alpha}, \end{aligned}$$

and finally the Orlicz–Besov norm

$$\|f\|_{B_{\Phi_\beta, \infty}^\alpha(I; X)} := \sup_{p \geq 1} p^{-1/\beta} \|f\|_{B_{p, \infty}^\alpha(I; X)} \approx \|f\|_{\mathfrak{L}^{\Phi_\beta}(I; X)} + \|f\|_{\Phi_\beta, \infty, \alpha}.$$

Because of the inequalities between different  $L^p$  norms, it is immediate that we have equivalent norms above, whether we understand  $p \geq 1$  as  $p \in [1, \infty)$  or  $p \in \{1, 2, \dots\}$ . For definiteness and later convenience, we choose the latter.

The above-given norm of  $\mathfrak{L}^{\Phi_\beta}(I; X)$  is equivalent to the usual norm of the Orlicz space  $L^{\Phi_\beta}(I; X)$  from Subsection 2.1 where  $\Phi_\beta(x) = \exp(|x|^\beta) - 1$  for  $\beta \geq 1$ . For  $0 < \beta < 1$ , the function  $\Phi_\beta$  must be defined in a slightly different way, but it is still essentially  $\exp(|x|^\beta)$ ; see [3].

For  $\beta \in \mathbb{Z}_+ \setminus \{0\}$  one can show in the same way as in [3, Theorem 3.4] that

$$(2.2) \quad \|f\|_{\mathfrak{L}^{\Phi_\beta}(I; X)} \leq \|f\|_{L^{\Phi_\beta}(I; X)}.$$

**2.4. Gaussian random variables.** Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space. As in [9] let  $X$  be Banach space with the following property: there exists a sequence  $(x_n^*)_{n \geq 1}$  in  $X^*$  such that  $\|x_n^*\| \leq 1$  and  $\|x\| = \sup_{n \geq 1} |x_n^*(x)|$ . Such a Banach space will be said to *admit a norming sequence of functionals*. Examples of such Banach spaces are all separable Banach spaces, but also spaces like  $l^\infty$ . As in [9] a mapping  $\xi : \Omega \rightarrow X$  will be called a *centered Gaussian* if for all  $x^* \in \text{span}\{x_n^* : n \geq 1\}$  the random variable  $\langle \xi, x^* \rangle$  is a centered Gaussian. For a centered Gaussian random variable we define

$$(2.3) \quad \sigma(\xi) = \sup_{n \geq 1} (\mathbb{E} |\langle \xi, x_n^* \rangle|^2)^{1/2}.$$

In [9] it is proved that

$$\lim_{t \rightarrow \infty} \frac{1}{t^2} \log \mathbb{P}(\|X\| > t) = -\frac{1}{2\sigma^2},$$

so that the value of  $\sigma$  is independent of the norming sequence  $(x_n^*)_{n \geq 1}$ .

We make some comment on the above definition of a Gaussian random variable. We do not assume that  $\xi$  is a Borel measurable mapping. The only obvious fact we will use is that the mapping  $\omega \mapsto \|\xi(\omega)\|$  is measurable. If  $\xi$  is a Gaussian random variable that takes values in a separable subspace of  $X$ , then  $\xi$  is Borel measurable and one already has that  $\langle \xi, x^* \rangle$  is a centered Gaussian random variables for all  $x^* \in X^*$ .

A random variable  $\xi : \Omega \rightarrow X$  is called *tight* if the measure  $\mathbb{P} \circ \xi^{-1}$  is tight, and it is called *Radon* if  $\mathbb{P} \circ \xi^{-1}$  is Radon. If  $X$  is a separable Banach space, then every Borel measurable random variable  $\xi : \Omega \rightarrow E$  is Radon, and in particular tight. Conversely, if a Gaussian random variable  $\xi : \Omega \rightarrow X$  is tight, then it almost surely takes values in a separable subspace of  $X$ . The next result is well-known and a short proof can be found in [9, p. 61].

**Proposition 2.1.** *Let  $X$  be a Banach space and let  $\xi : \Omega \rightarrow X$  be a centered Gaussian. If  $\xi$  is tight, then  $\mathbb{P}(\|\xi\| < r) > 0$  for all  $r > 0$ .*

### 3. MAXIMAL ESTIMATES FOR SEQUENCES OF GAUSSIAN RANDOM VARIABLES

The next proposition together with Theorem 3.1 may be considered as the vector-valued extension of a result in [4].

**Proposition 3.1.** *Let  $X$  be a Banach space which admits a norming sequence of functionals  $(x_n^*)_{n \geq 1}$ . Let  $\Theta$  be as in (2.1). Let  $(\xi_n)_{n \geq 1}$  be  $X$ -valued centered Gaussian random variables with first moments and weak variances*

$$m_n = \mathbb{E}\|\xi_n\|,$$

$$\sigma_n = \sup_{m \geq 1} (\mathbb{E}|\langle \xi_n, x_m^* \rangle|^2)^{1/2}.$$

*It holds that*

$$\mathbb{E} \sup_{n \geq 1} \|\xi_n\| \leq m + 3\rho_\Theta((\sigma_n)_{n \geq 1}),$$

*where  $m = \sup_{n \geq 1} m_n$ .*

*Moreover, if any linear combination of the  $(\xi_n)_{n \geq 1}$  is a Gaussian random variable and if  $\mathbb{E} \sup_{n \geq 1} \|\xi_n\| < \infty$ , then  $\xi := (\xi_n)_{n \geq 1}$  is an  $l^\infty(X)$ -valued Gaussian random variable.*

By the Kahane-Khinchine inequalities (cf. [8, Corollary 3.4.1]) one obtains a similar estimate for the  $p$ -th moments of  $\sup_{n \geq 1} \|\xi_n\|$ . However, this also follows by extending the proof below.

*Proof.* We may write

$$\mathbb{E} \sup_{n \geq 1} \|\xi_n\| \leq \mathbb{E} \sup_{n \geq 1} |\|\xi_n\| - m_n| + \sup_{n \geq 1} m_n$$

By [9, (3.2)] for all  $t > 0$ , we have

$$(3.1) \quad \mathbb{P}(|\|\xi_n\| - m_n| > t) \leq 2 \exp\left(-\frac{t^2}{2\sigma_n^2}\right).$$

For each  $\delta > 0$  it follows that

$$\begin{aligned}
 \mathbb{E} \sup_{n \geq 1} |\|\xi_n\| - m_n| &= \int_0^\infty \mathbb{P}(\sup_{n \geq 1} |\|\xi_n\| - m_n| > t) dt \\
 &\leq \delta + \int_\delta^\infty \mathbb{P}(\sup_{n \geq 1} |\|\xi_n\| - m_n| > t) dt \leq \delta + \sum_{n \geq 1} \int_\delta^\infty \mathbb{P}(|\|\xi_n\| - m_n| > t) dt \\
 (3.2) \quad &\leq \delta + \sum_{n \geq 1} 2 \int_\delta^\infty \exp\left(-\frac{t^2}{2\sigma_n^2}\right) dt = \delta + \sum_{n \geq 1} 2 \int_{\delta/\sigma_n}^\infty \sigma_n \exp\left(-\frac{t^2}{2}\right) dt \\
 &\leq \delta + 2 \sum_{n \geq 1} \frac{\sigma_n^2}{\delta} \exp\left(-\frac{\delta^2}{2\sigma_n^2}\right) = \delta \left[1 + 2 \sum_{n \geq 1} \frac{\sigma_n^2}{\delta^2} \exp\left(-\frac{\delta^2}{2\sigma_n^2}\right)\right]
 \end{aligned}$$

where we used the standard estimate

$$\int_\delta^\infty e^{-t^2/2} dt \leq \frac{1}{\delta} \exp(-\delta^2/2).$$

If  $\delta > 0$  is chosen so that the last series sums up to at most 1, then we have shown that  $\mathbb{E} \sup_{n \geq 1} |\|\xi_n\| - m_n| \leq 3\delta$ . Taking the infimum over all such  $\delta$ , we obtain the result.

The final assertion follows from the definition of a Gaussian random variable using the norming sequence of functionals  $(e_m \otimes x_n^*)_{m,n \geq 1}$ .  $\square$

*Remark 3.1.* The infimum appearing in Proposition 3.1 is dominated by

$$\left[ \left( \frac{p-1}{e} \right)^{\frac{p-1}{2}} \sum_{n \geq 1} \sigma_n^{p+1} \right]^{1/(p+1)}$$

for any  $p \in [1, \infty[$ . (Interpret  $0^0 = 1$  for  $p = 1$ .) This follows from the elementary estimate  $e^{-x^2/2} \leq [(p-1)/e]^{(p-1)/2} x^{1-p}$  applied to  $x = \delta/\sigma_n$ .

For an  $X$ -valued random variable  $\xi$  we take a median  $M$  such that

$$\mathbb{P}(\|\xi\| \leq M) \geq 1/2 \quad \text{and} \quad \mathbb{P}(\|\xi\| \geq M) \geq 1/2.$$

For convenience we will take  $M = M(\xi)$  to be the smallest possible  $M$ . Notice that for all  $p \in (0, \infty)$ ,  $\mathbb{E}\|\xi\|^p \geq \frac{M^p}{2}$ .

Alternatively, we could have replaced the estimate (3.1) in the above proof by

$$\mathbb{P}(|\|\xi\| - M| > t) \leq \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

(see [9, Lemma 3.1]) to obtain:

**Proposition 3.2.** *Let  $X$  be a Banach space which admits a norming sequence of functionals  $(x_n^*)_{n \geq 1}$ . Let  $\Theta$  be as in (2.1). Let  $(\xi_n)_{n \geq 1}$  be  $X$ -valued centered Gaussian random variables with medians  $M_n$  and weak variances*

$$\sigma_n = \sup_{m \geq 1} (\mathbb{E} |\langle \xi_n, x_m^* \rangle|^2)^{1/2}.$$

*It holds that*

$$\mathbb{E} \sup_{n \geq 1} \|\xi_n\| \leq M + 2\rho_\Theta((\sigma_n)_{n \geq 1}),$$

*where  $M = \sup_{n \geq 1} M_n$ .*

If the  $\xi_n$  are independent Gaussian random variables, then a converse to Proposition 3.1 holds.

**Theorem 3.1.** *Let  $X$  be a Banach space which admits a norming sequence of functionals. Let  $\Theta$  be as in (2.1). Let  $(\xi_n)_{n \geq 1}$  be  $X$ -valued independent centered Gaussian random variables with first moments  $(m_n)_{n \geq 1}$  and weak variances  $(\sigma_n)_{n \geq 1}$ . Let  $m = \sup_{n \geq 1} m_n$ . It holds that*

$$\begin{aligned} \mathbb{E} \sup_{n \geq 1} \|\xi_n\| &\approx m + \rho_\Theta((\sigma_n)_{n \geq 1}) \\ &\approx m + \|(\sigma_n)_{n \geq 1}\|_\Theta. \end{aligned}$$

Moreover, if one of these expressions is finite, then  $\xi := (\xi_n)_{n \geq 1}$  is an  $l^\infty(X)$ -valued Gaussian random variable.

Recall from Subsection 2.1 and the definition of  $\Theta$  that

$$\|(\sigma_n)_{n \geq 1}\|_\Theta = \inf_{\delta > 0} \left\{ \frac{1}{\delta} \left[ 1 + \sum_{n \geq 1} \delta^2 \sigma_n^2 \exp \left( -\frac{1}{2\delta^2 \sigma_n^2} \right) \right] \right\}.$$

*Proof.* The second two sided estimate follows from Lemma 2.1.

The estimate  $\lesssim$  in the first comparison has been obtained in Proposition 3.1. To prove  $\gtrsim$ , note that  $\mathbb{E} \sup_{n \geq 1} \|\xi_n\| \geq m$  is clear. As for the estimate for  $\rho_\Theta((\sigma_n)_{n \geq 1})$ , by scaling we may assume that  $\mathbb{E} \sup_{n \geq 1} \|\xi_n\| = 1$ . Then one has  $\mathbb{P}(\sup_{n \geq 1} \|\xi_n\| > 3) \leq 1/3$ , and therefore

$$\begin{aligned} 1/3 &\leq \mathbb{P}(\sup_{n \geq 1} \|\xi_n\| \leq 3) = \prod_{n \geq 1} \mathbb{P}(\|\xi_n\| \leq 3) = \prod_{n \geq 1} (1 - \mathbb{P}(\|\xi_n\| > 3)) \\ &\leq \prod_{n \geq 1} \exp \left( -\mathbb{P}(\|\xi_n\| > 3) \right). \end{aligned}$$

It follows that

$$\log 3 \geq \sum_{n \geq 1} \mathbb{P}(\|\xi_n\| > 3).$$

Let  $\varepsilon \in (0, 1)$  be an arbitrary number. If for each  $n \geq 1$ , we choose  $k_n$  such that  $(\mathbb{E} \langle \xi_n, x_{k_n}^* \rangle^2)^{1/2} \geq \sigma_n(1 - \varepsilon)$ , then we obtain

$$\begin{aligned} \log 3 &\geq \sum_{n \geq 1} \mathbb{P}(\|\xi_n\| > 3) \geq \sum_{n \geq 1} \mathbb{P}(|\langle \xi_n, x_{k_n}^* \rangle| > 3) \\ &\geq \sqrt{\frac{2}{\pi}} \sum_{n \geq 1} \frac{3\sigma_n(1 - \varepsilon)}{\sigma_n^2(1 - \varepsilon)^2 + 9} \exp \left( -\frac{9}{2\sigma_n^2(1 - \varepsilon)^2} \right) \end{aligned}$$

where we used  $\int_a^\infty e^{-t^2/2} dt \geq \frac{a}{1+a^2} e^{-a^2/2}$ . Next, we have

$$\sigma_n^2 = \sup_{m \geq 1} \mathbb{E} \langle \xi_n, x_m^* \rangle^2 = \frac{\pi}{2} \sup_{m \geq 1} \mathbb{E} |\langle \xi_n, x_m^* \rangle| \leq \frac{\pi}{2} \mathbb{E} \|\xi_n\| \leq \frac{\pi}{2},$$

hence  $\sigma_n^2(1 - \varepsilon)^2 + 9 \leq \pi/2 + 9 < 11$  and  $\sqrt{2/\pi} \cdot \sigma_n \geq 2/\pi \cdot \sigma_n^2$ , thus

$$\log 3 \geq \frac{6}{11\pi} \sum_{n \geq 1} \sigma_n^2(1 - \varepsilon) \exp \left( -\frac{9}{2\sigma_n^2(1 - \varepsilon)^2} \right).$$



This being true for all  $\varepsilon > 0$ , it follows in the limit that

$$\sum_{n \geq 1} \left( \frac{\sigma_n}{3} \right)^2 \exp \left( - \frac{9}{2\sigma_n^2} \right) \leq \log 3 \cdot \frac{11\pi}{6 \cdot 9} < 1.$$

Therefore,  $\rho_\Theta((\sigma_n)_{n \geq 1}) \leq 3$ .

The last assertion follows as in Proposition 3.1.  $\square$

From the proof of Theorem 3.1 we actually see that

$$\mathbb{E} \sup_{n \geq 1} \|\xi_n\| \geq \max \left\{ \frac{1}{3} \rho_\Theta((\sigma_n)_{n \geq 1}), m \right\}.$$

*Remark 3.2.* A similar proof as presented above shows that the function  $\Theta$  in Theorem 3.1 can be replaced by the (non-convex) function  $\Phi$  defined below Lemma 2.1. Since we prefer to have an Orlicz space, we use the convex function  $\Theta$ .

In the real-valued case,  $m$  is not needed in the estimate of Theorem 3.1. This is due to the fact that it can be estimated by  $\sup_{n \geq 1} \sigma_n$ . The following simple example shows that in the infinite dimensional setting this is not the case. We shall also encounter the same phenomenon in a more serious example in the proof of Theorem 6.1.

*Example 3.1.* Let  $p \in [1, \infty]$  and let  $X = l^p$  with the standard unit vectors denoted by  $e_n$ . Let  $(\sigma_n)_{n \geq 1}$  be a sequence of positive real numbers with

$$m_p := \left( \sum_{n \geq 1} \sigma_n^p \right)^{1/p} < \infty \quad \text{if } p < \infty$$

and

$$m_\infty := \rho_\Theta((\sigma_n)_{n \geq 1}) < \infty \quad \text{if } p = \infty.$$

Let  $(\gamma_n)_{n \geq 1}$  be a sequence of independent standard Gaussian random variables. Then  $\xi = \sum_{n \geq 1} \sigma_n \gamma_n e_n$  defines an  $X$ -valued Gaussian random variable with  $m(\xi) = \mathbb{E}\|\xi\| \approx_p m_p$  and

$$\sigma(\xi) = \begin{cases} \sup_{n \geq 1} \sigma_n & p \in [2, \infty] \\ \left( \sum_{n \geq 1} \sigma_n^r \right)^{\frac{1}{r}} & p \in [1, 2), \end{cases}$$

where  $r = \frac{2p}{2-p}$ .

#### 4. BESOV REGULARITY OF BROWNIAN PATHS

We say that an  $X$ -valued process  $(W(t))_{t \in [0,1]}$  is a *Brownian motion* if it is strongly measurable and for all  $x^* \in E^*$ ,  $(\langle W(t), x^* \rangle)_{t \in [0,1]}$  is a real Brownian motion starting at zero. Let  $Q$  be the covariance of  $W(1)$ . For the process  $W$  we have

- (1)  $W(0) = 0$ ,
- (2)  $W$  has a version with continuous paths,
- (3)  $W$  has independent increments,
- (4) For all  $0 \leq s < t < \infty$ ,  $W(t) - W(s)$  has distribution  $\mathcal{N}(0, (t-s)Q)$ .

In this situation we say that  $W$  is a Brownian motion *with covariance*  $Q$ . Notice that every process  $W$  that satisfies (3) and (4) has a path-wise continuous version (cf. [5, Theorem 3.23]).

In the next result we obtain a Besov regularity result for Brownian motions. The case of real valued Brownian motions has been considered in [2, 3, 11]. But even in the real-valued case we believe the proof is new and more direct.

**Theorem 4.1.** *Let  $X$  be a Banach space and let  $p, q \in [1, \infty)$ . For an  $X$ -valued non-zero Brownian motion  $W$  we have*

$$\begin{aligned} W &\in B_{\Phi_2, \infty}^{1/2}(0, 1; X) \subset B_{p, \infty}^{1/2}(0, 1; X) && a.s., \\ W &\notin B_{p, q}^{1/2}(0, 1; X) && a.s. \end{aligned}$$

*Proof.* Denote

$$Y_{n,p} := 2^{n/2} \|W(\cdot + 2^{-n}) - W\|_{L^p(I(2^{-n}); X)}.$$

We may write

$$\begin{aligned} Y_{n,p}^p &= \int_0^{1-2^{-n}} 2^{np/2} \|W(t+2^{-n}) - W(t)\|^p dt \\ &= \sum_{m=1}^{2^n-1} \int_{(m-1)2^{-n}}^{m2^{-n}} 2^{np/2} \|W(t+2^{-n}) - W(t)\|^p dt \\ &= \sum_{m=1}^{2^n-1} 2^{-n} \int_0^1 2^{np/2} \|W((s+m)2^{-n}) - W((s+m-1)2^{-n})\|^p ds \\ &= \int_0^1 2^{-n} \sum_{m=1}^{2^n-1} \|\gamma_{n,m,s}\|^p ds \end{aligned}$$

Here  $\gamma_{n,m,s} = 2^{n/2}(W((s+m)2^{-n}) - W((s+m-1)2^{-n}))$ . For fixed  $s \in (0, 1)$  and  $n \geq 1$ ,  $(\gamma_{n,m,s})_{m \geq 1}$  is a sequence of independent random variables distributed as  $W(1)$ . Denote  $c_p = (\mathbb{E}\|W(1)\|^p)^{1/p}$ . If we take second moments we may use Jensen's inequality to obtain

$$\begin{aligned} \mathbb{E}(Y_{n,p}^p - c_p^p)^2 &= \mathbb{E} \left| \int_0^1 [2^{-n} \sum_{m=1}^{2^n-1} \|\gamma_{n,m,s}\|^p - c_p^p] ds \right|^2 \\ &\leq \int_0^1 \mathbb{E} \left| 2^{-n} \sum_{m=1}^{2^n-1} (\|\gamma_{n,m,s}\|^p - c_p^p) - 2^{-n} c_p^p \right|^2 ds \\ &= \int_0^1 \left[ 2^{-2n} (2^n - 1) (c_{2p}^{2p} - c_p^{2p}) + 2^{-2n} c_p^{2p} \right] ds \\ &= 2^{-n} [(1 - 2^{-n}) c_{2p}^{2p} - (1 - 2^{1-n}) c_p^{2p}]. \end{aligned}$$

It follows that for a fixed  $\varepsilon > 0$ , we have

$$\sum_{n \geq 1} \mathbb{P}(|Y_{n,p}^p - c_p^p| > \varepsilon) \leq \frac{1}{\varepsilon^2} \sum_{n \geq 1} \mathbb{E}(Y_{n,p}^p - c_p^p)^2 < \infty,$$

which implies, by the Borel–Cantelli Lemma, that

$$\mathbb{P}\left(|Y_{n,p}^p - c_p^p| > \varepsilon \text{ infinitely often}\right) = 0.$$

This in turn gives that

$$(4.1) \quad \lim_{n \rightarrow \infty} 2^{n/2} \|W(\cdot + 2^{-n}) - W\|_{L^p(I(2^{-n}); X)} = (\mathbb{E} \|W(1)\|^p)^{1/p} \text{ a.s.}$$

This shows immediately that the paths are a.s. in  $B_{p,\infty}^{1/2}(0, 1; X)$ . From the above calculation it is also clear that  $W \notin B_{p,q}^{1/2}(0, 1; X)$  a.s. for  $q \in [1, \infty)$ . Next we show that the paths are in  $B_{\Phi_2,\infty}^{1/2}(0, 1; X)$  a.s. Note that  $(\mathbb{E} \|W(1)\|^p)^{1/p} \approx p^{1/2}$  as  $p \rightarrow \infty$ . The upper estimate  $\lesssim$  is a consequence of Fernique’s theorem (which says that  $\|W(1)\|^2$  is exponentially integrable, since  $W(1)$  is a non-zero  $X$ -valued Gaussian random variable), whereas  $\gtrsim$  follows from the corresponding estimate for real Gaussians after applying a functional. We proved that  $\mathbb{E}(Y_{n,p}^p - c_p^p)^2 \leq c_{2p}^{2p} 2^{-n}$ . Therefore,

$$\mathbb{E}(Y_{n,p}^p c_p^{-p} - 1)^2 \leq 2^{-n} c_{2p}^{2p} c_p^{-2p} \leq 2^{-n} K^{2p},$$

where  $K \geq 1$  is some constant. Hence for all  $\lambda > 1$ ,

$$\mathbb{P}(Y_{n,p}^p c_p^{-1} > \lambda) \leq \mathbb{P}(|Y_{n,p}^p c_p^{-p} - 1| > \lambda^p - 1) \leq 2^{-n} K^{2p} (\lambda^p - 1)^{-2},$$

and thus for  $\lambda = 2K$

$$\sum_{n,p=1}^{\infty} \mathbb{P}(Y_{n,p}^p c_p^{-1} > \lambda) \leq \sum_{n=1}^{\infty} 2^{-n} \sum_{p=1}^{\infty} K^{2p} (\lambda^p - 1)^{-2} < \infty$$

so that by the Borel–Cantelli lemma

$$\mathbb{P}\left(Y_{n,p}^p c_p^{-1} > \lambda \text{ for infinitely many pairs } (n, p)\right) = 0.$$

Since  $c_p \approx p^{1/2}$  this means that a.s.

$$\sup_{n,p} 2^{n/2} \|W(\cdot + 2^{-n}) - W\|_{L^p(I(2^{-n}); X)} p^{-1/2} < \infty.$$

□

## 5. BROWNIAN MOTIONS AS RANDOM VARIABLES IN BESOV SPACES

From the pathwise properties of  $W$  studied in the previous section, we know that we have a function  $W : \Omega \rightarrow B_{p,\infty}^{1/2}$ . We now go into the measurability issues in order to promote it to a random variable.

**Theorem 5.1.** *Let  $X$  be a Banach space and let  $p \in [1, \infty)$ . Then an  $X$ -valued Brownian motion  $W$  is a  $B_{p,\infty}^{1/2}(0, 1; X)$ -valued, and even  $B_{\Phi_2,\infty}^{1/2}(0, 1; X)$ -valued, Gaussian random variable. In particular, there exists an  $\varepsilon > 0$  such that*

$$\mathbb{E} \exp(\varepsilon \|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)}^2) < \infty.$$

*If  $W$  is non-zero, then the random variables  $W : \Omega \rightarrow B_{p,\infty}^{1/2}(0, 1; X)$  and  $W : \Omega \rightarrow B_{\Phi_2,\infty}^{1/2}(0, 1; X)$  are not tight. In fact,*

$$\tau_1 := \inf\{\lambda \geq 0 : \mathbb{P}(\|W\|_{B_{p,\infty}^{1/2}(0,1;X)} \leq \lambda) > 0\} \geq (\mathbb{E} \|W(1)\|^p)^{1/p},$$

*and consequently also*

$$\tau_2 := \inf\{\lambda \geq 0 : \mathbb{P}(\|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)} \leq \lambda) > 0\} > 0.$$

There is some interest in the numbers  $\tau_1$  and  $\tau_2$ . For general theory we refer the reader to [9, Chapter 3].

For the proof we need the following easy lemma.

**Lemma 5.1.** *Let  $X$  be a Banach space which admits a norming sequence, let  $0 < \alpha < 1$  and  $0 < \beta < \infty$ . Then for all  $p \in [1, \infty)$  there exist*

$$(\Lambda_{pjk})_{j \geq 0, k \geq 1} \subset B_{p, \infty}^\alpha(0, 1; X)^* \subset B_{\Phi_\beta, \infty}^\alpha(0, 1; X)^*,$$

$$(f_{pjk})_{j \geq 0, k \geq 1} \subset C^\infty([0, 1]; X^*),$$

such that: for all  $\phi \in B_{p, \infty}^\alpha(0, 1; X)$  there are the representations

$$\langle \phi, \Lambda_{p0k} \rangle = \int_0^1 \langle \phi(t), f_{p0k}(t) \rangle dt, \quad k \geq 1,$$

$$\langle \phi, \Lambda_{pjk} \rangle = \int_0^{1-2^{-j}} 2^{j\alpha} \langle \phi(t + 2^{-j}) - \phi(t), f_{pjk}(t) \rangle dt, \quad j, k \geq 1;$$

we have the upper norm bounds

$$p^{-1/\beta} \|\Lambda_{pjk}\|_{B_{\Phi_\beta, \infty}^\alpha(0, 1; X)^*} \leq \|\Lambda_{pjk}\|_{B_{p, \infty}^\alpha(0, 1; X)^*} \leq 1, \quad k \geq 1;$$

and finally the sequences are norming in the following sense:

$$\begin{aligned} \|\phi\|_{B_{p, \infty}^\alpha(0, 1; X)} &= \sup_{j \geq 0, k \geq 1} |\langle \phi, \Lambda_{pjk} \rangle|, \\ \|\phi\|_{B_{\Phi_\beta, \infty}^\alpha(0, 1; X)} &= \sup_{p \geq 1, j \geq 0, k \geq 1} p^{-1/\beta} |\langle \phi, \Lambda_{pjk} \rangle|. \end{aligned}$$

*Proof.* Let  $(x_n^*)_{n \geq 1}$  be a norming sequence for  $X$ . Let  $I = [a, b]$ . First observe that there exists a sequence  $(F_k)_{k \geq 1}$  in  $L^{p'}(I; X^*)$ , with norm smaller than or equal to one, which is norming for  $L^p(I; X)$ . Such a sequence is easily constructed using the  $(x_n^*)_{n \geq 1}$  and standard duality arguments. By an approximation argument we can even take the  $(F_k)_{k \geq 1}$  in  $C^\infty(I; X^*)$ .

To prove the lemma, let first  $a = 0$  and  $b = 1$ , and let  $(f_{p0k})_{k \geq 1}$  be the above constructed sequence  $(F_k)_{k \geq 1}$ . Next we fix  $j \geq 1$  and let  $a = 0$  and  $b = 1 - 2^{-j+1}$  and let  $(f_{pjk})_{k \geq 1}$  be the above constructed sequence for this interval. Let  $\Lambda_{pjk}$  be the elements in  $B_{p, \infty}^\alpha(0, 1; X)^*$  defined as in the statement in the lemma. It is easily checked that this sequence satisfies the required properties.  $\square$

*Proof.* Proof of Theorem 5.1 Since  $W$  is strongly measurable as an  $X$ -valued process we may assume that  $X$  is separable and therefore that it admits a norming sequence. In Theorem 4.1 it has been shown that the paths of  $W$  are a.s. in  $B_{\Phi_2, \infty}^{1/2}(0, 1; X) \subset B_{p, \infty}^{1/2}(0, 1; X)$  for all  $p \in [1, \infty)$ . It follows from Lemma 5.1 that there exists a norming sequence of functionals  $(\Lambda_n)_{n \geq 1}$  for  $B_{\Phi_2, \infty}^{1/2}(0, 1; X)$ , as well as in each  $B_{p, \infty}^{1/2}(0, 1; X)$ , such that  $\langle W, \Lambda \rangle$  is a centered Gaussian random variable for all  $\Lambda \in \text{span}\{\Lambda_n, n \geq 1\}$ . Therefore, by definition it follows that  $W$  is a centered Gaussian random variable. The exponential integrability follows from [9, Corollary 3.2].

The last assertion follows from (4.1). This also shows that  $W$  is not tight since, by Proposition 2.1, for centered Gaussian measures which are tight, one has  $\tau = 0$ .  $\square$

## 6. MOMENT ESTIMATES FOR BROWNIAN MOTIONS IN BESOV SPACES

Now that we know that

$$\mathbb{E}\|W\|_{B_{p,\infty}^{1/2}(0,1;X)} < \infty, \quad \mathbb{E}\|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)} < \infty,$$

it seems interesting to estimate these quantities. For this we need a convenient representation of  $X$ -valued Brownian motions.

Recall that a family  $W_H = (W_H(t))_{t \in \mathbb{R}_+}$  of bounded linear operators from  $H$  to  $L^2(\Omega)$  is called an  $H$ -cylindrical Brownian motion if

- (1)  $W_H h = (W_H(t)h)_{t \in \mathbb{R}_+}$  is a real-valued Brownian motion for each  $h \in H$ ,
- (2)  $\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s \wedge t) [g, h]_H$  for all  $s, t \in \mathbb{R}_+$ ,  $g, h \in H$ .

We always assume that the  $H$ -cylindrical Brownian motion  $W_H$  is adapted to a given filtration  $\mathcal{F}$ , i.e., the Brownian motions  $W_H h$  are adapted to  $\mathcal{F}$  for all  $h \in H$ . Notice that if  $(h_n)_{n \geq 1}$  is an orthonormal basis for  $H$ , then  $(W_H h_n)_{n \geq 1}$  are independent standard real-valued Brownian motions.

Let  $W : \mathbb{R}_+ \times \Omega \rightarrow E$  be an  $E$ -valued Brownian motion and let  $Q \in \mathcal{L}(E^*, E)$  be its covariance operator. Let  $H_Q$  be the reproducing kernel Hilbert space or Cameron–Martin space (cf. [1, 13]) associated with  $Q$  and let  $i_W : H_Q \hookrightarrow E$  be the inclusion operator. Then the mappings

$$W_{H_Q}(t) : i_W^* x^* \mapsto \langle W(t), x^* \rangle$$

uniquely extend to an  $H_Q$ -cylindrical Brownian motion  $W_{H_Q}$ , so that in particular

$$(6.1) \quad \langle W(t), x^* \rangle = W_{H_Q}(t) i_W^* x^*.$$

**Lemma 6.1.** *There holds, for all  $p \in [1, \infty)$ ,*

$$\|i_W\| = \sigma(W(1)) \lesssim \frac{1}{\sqrt{p}} (\mathbb{E}\|W(1)\|^p)^{1/p}.$$

*Proof.* Note first that, since  $\langle W(t), x^* \rangle$  is a real-valued Gaussian random variable, its moments satisfy

$$(6.2) \quad (\mathbb{E}|\langle W(t), x^* \rangle|^p)^{1/p} = \gamma_p (\mathbb{E}|\langle W(t), x^* \rangle|^2)^{1/2},$$

where the  $\gamma_p$  are universal constants behaving like  $\gamma_p \approx \sqrt{p}$  for  $p \in [1, \infty)$ .

On the other hand, by (6.1) and the definition of cylindrical Brownian motion,

$$(\mathbb{E}|\langle W(t), x^* \rangle|^2)^{1/2} = \sqrt{t} \|i_W^* x^*\|.$$

With  $t = 1$ , taking supremum over all  $x^* \in X^*$  of unit norm, and recalling that  $\|i_W\| = \|i_W^*\|$ , this proves the first equality in the assertion. The second then follows from (6.2) and the obvious estimate

$$(\mathbb{E}|\langle W(t), x^* \rangle|^p)^{1/p} \leq (\mathbb{E}\|W(t)\|^p)^{1/p}$$

for  $\|x^*\| \leq 1$ . □

**Lemma 6.2.** *Let  $c > 0$ , and  $J \subset \mathbb{R}_+$  be an interval of length  $|J| \geq c$ . Consider  $W(\cdot + c) - W$  as an  $L^p(J, X)$ -valued Gaussian random variable. Then*

$$\sigma(W(\cdot + c) - W) \approx c^{1/2+1/p} \|i_W\|.$$

*Proof.* To prove the claim take  $f \in L^{p'}(J; X^*)$ . We also use the same symbol for its extension to  $\mathbb{R}$  with zero fill. The representation (6.1), the Stochastic Fubini theorem, and the Itô isometry yield

$$\begin{aligned}
& \left( \mathbb{E} \left| \int_J \langle (W(t+c) - W(t)), f(t) \rangle dt \right|^2 \right)^{1/2} \\
&= \left( \mathbb{E} \left| \int_J (W_H(t+c) - W_H(t)) i_W^* f(t) dt \right|^2 \right)^{1/2} \\
&= \left( \mathbb{E} \left| \int_{\mathbb{R}} \int_{\mathbb{R}_+} \mathbf{1}_{[t, t+c]}(s) i_W^* f(t) dW_H(s) dt \right|^2 \right)^{1/2} \\
&= \left( \mathbb{E} \left| \int_{\mathbb{R}_+} \mathbf{1}_{[0, c]} * (i_W^* f)(s) dW_H(s) \right|^2 \right)^{1/2} \\
&= \left( \int_{\mathbb{R}} \|\mathbf{1}_{[0, c]} * (i_W^* f)(s)\|_H^2 ds \right)^{1/2}.
\end{aligned}$$

Taking the supremum over all  $f \in L^{p'}(J; X^*)$  of unit norm, we find that

$$\sigma(W(\cdot + c) - W) = \|(\mathbf{1}_{[0, c]} *) \otimes i_W^*\|_{L^{p'}(J; X^*) \rightarrow L^2(\mathbb{R}; H)}.$$

By Young's inequality with  $1 + 1/2 = 1/p' + 1/r$  it follows that the operator norm is dominated by

$$\|\mathbf{1}_{[0, c]}\|_{L^r} \|i_W^*\|_{X^* \rightarrow H} = c^{1/p+1/2} \|i_W\|.$$

On the other hand, if we test with the functions  $f = \mathbf{1}_I \otimes x^* \in L^{p'}(J; X^*)$ , where  $I \subseteq J$  has length  $c$ , we find that

$$\begin{aligned}
\|\mathbf{1}_{[0, c]} * (i_W^* f)\|_{L^2(H)} &= \|\mathbf{1}_{[0, c]} * \mathbf{1}_I\|_{L^2} \|i_W^* x^*\|_H \\
&= (2/3)^{1/2} c^{3/2} \|i_W^* x^*\|_H \approx c^{1/2+1/p} \frac{\|i_W^* x^*\|_H}{\|x^*\|_{X^*}} \|f\|_{L^{p'}(X^*)}.
\end{aligned}$$

Taking the supremum over  $x^* \in X^* \setminus \{0\}$  we get the other side of the asserted norm equivalence.  $\square$

**Corollary 6.1.** *Let  $c \in (0, e^{-1/2}]$ , and  $J \subset \mathbb{R}_+$  be an interval of length  $|J| \geq c$ . Consider  $W(\cdot + c) - W$  as an  $\mathfrak{L}^{\Phi_2}(J; X)$ -valued Gaussian random variable. Then*

$$\sigma(W(\cdot + c) - W) \approx (\log c^{-1})^{-1/2} c^{1/2} \|i_W\|.$$

*Proof.* We note that the functionals  $p^{-1/2} \Lambda_{p0k}$  from Lemma 5.1 (with  $\beta = 2$ ) provide a norming sequence for  $\mathfrak{L}^{\Phi_2}(0, 1; X)$ , and the same construction can be adapted to another interval. Hence

$$\begin{aligned}
& \sigma_{\mathfrak{L}^{\Phi_2}(J; X)}(W(\cdot + c) - W) \\
&= \sup_{p \geq 1} p^{-1/2} \sup_{k \geq 1} \left( \mathbb{E} \left| \int_J \langle (W(t+c) - W(t)), f_{p0k}(t) \rangle dt \right|^2 \right)^{1/2} \\
&= \sup_{p \geq 1} p^{-1/2} \sigma_{L^p(J; X)}(W(\cdot + c) - W) \\
&\approx \sup_{p \geq 1} p^{-1/2} c^{1/2+1/p} \|i_W\| \\
&\approx (\log c^{-1})^{-1/2} c^{1/2} \|i_W\|,
\end{aligned}$$

where an elementary maximum value problem was solved in the last step.  $\square$

**Theorem 6.1.** *Let  $X$  be a Banach space. Let  $p \in [1, \infty)$ . For an  $X$ -valued Brownian motion  $W$  we have*

$$(6.3) \quad \mathbb{E}\|W\|_{B_{p,\infty}^{1/2}(0,1;X)} \approx (\mathbb{E}\|W(1)\|^p)^{1/p},$$

$$(6.4) \quad \mathbb{E}\|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)} \approx \mathbb{E}\|W(1)\|.$$

*Remark 6.1.* By [9, Corollary 3.2], the estimate (6.3) implies that

$$\mathbb{E}\|W\|_{B_{p,\infty}^{1/2}(0,1;X)} \lesssim \sqrt{p} \mathbb{E}\|W(1)\|,$$

but we do not know if there is a two sided comparison here. The above estimate is also an immediate consequence of (6.4) and the definition of the various norms.

*Proof.* As in Theorem 5.1 we may assume that  $X$  admits a norming sequence.

The estimate  $\gtrsim$  in (6.3) follows from (4.1). Let us then consider the other direction. Clearly,

$$\mathbb{E}\|W\|_{L^p(0,1;X)} \leq (\mathbb{E}\|W\|_{L^\infty(0,1;X)}^2)^{1/2} \leq 2(\mathbb{E}\|W(1)\|^2)^{1/2} \lesssim \mathbb{E}\|W(1)\|$$

by Doob's maximal inequality and the equivalence of Gaussian moments. Next we consider

$$(6.5) \quad \mathbb{E} \sup_{j \geq 1} 2^{j/2} \|W(\cdot + 2^{-j}) - W\|_{L^p(0,1-2^{-j};X)}.$$

This can be estimated using Proposition 3.1 with the  $L^p(0,1;X)$ -valued Gaussian random variables  $\xi_j = 2^{j/2}[W(\cdot + 2^{-j}) - W]1_{[0,1-2^{-j}]}$ :

$$\mathbb{E} \sup_{j \geq 1} \|\xi_j\| \lesssim \sup_{j \geq 1} \mathbb{E}\|\xi_j\| + \|(\sigma_j)_{j \geq 1}\|_\Theta.$$

The first term is clearly smaller than  $(\mathbb{E}\|W(1)\|^p)^{1/p}$ . By Lemma 6.2 and Example 2.1, the Orlicz norm can be computed as

$$\begin{aligned} \|(\sigma_j)_{j \geq 1}\|_\Theta &\approx \|i_W\| \|(2^{-j/p})_{j \geq 1}\|_\Theta \approx \|i_W\| \sqrt{\log(1 - 2^{-1/p})^{-1}} \\ &\approx (1 + \sqrt{\log p}) \|i_W\|. \end{aligned}$$

By Lemma 6.1, this is smaller than  $(\mathbb{E}\|W(1)\|^p)^{1/p}$ ; indeed, it is much smaller when  $p \rightarrow \infty$ . Thus, just like in Example 3.1, we are in a situation where the  $m$  term totally dominates in the estimate (1.1). The proof of (6.3) is complete.

Next, we show (6.4). The lower estimate follows trivially from (6.3). For the upper estimate we write

$$\begin{aligned} &\mathbb{E}\|W\|_{B_{\Phi_2,\infty}^{1/2}(0,1;X)} \\ &\leq \mathbb{E}\|W\|_{\mathfrak{L}^{\Phi_2}(0,1;X)} + \mathbb{E} \sup_{j \geq 1} 2^{j/2} \|W(\cdot + 2^{-j}) - W\|_{\mathfrak{L}^{\Phi_2}(0,1-2^{-j};X)}. \end{aligned}$$

The first term can again be estimated using Doob's maximal inequality, since

$$\mathbb{E}\|W\|_{\mathfrak{L}^{\Phi_2}(0,1;X)} \leq \mathbb{E}\|W\|_{L^\infty(0,1;X)}.$$

The second term can be treated using Proposition 3.1 with the  $\mathfrak{L}^{\Phi_2}(0,1;X)$ -valued Gaussian random variables  $\xi_j = 2^{j/2}[W(\cdot + 2^{-j}) - W]1_{[0,1-2^{-j}]}$ . Combining that Proposition with Remark 3.1, we have

$$\mathbb{E} \sup_{j \geq 1} \|\xi_j\| \lesssim \sup_{j \geq 1} \mathbb{E}\|\xi_j\| + \left( \sum_{j \geq 1} \sigma_j^4 \right)^{1/4}.$$

From Corollary 6.1 we get

$$\sigma_j \lesssim (\log 2^j)^{-1/2} \|i_W\| \approx j^{-1/2} \|i_W\|,$$

so that the series sums up to  $\lesssim \|i_W\| \lesssim \mathbb{E}\|W(1)\|$ .

We then estimate  $\mathbb{E}\|\xi_j\|$ . By (2.2), we have

$$\begin{aligned} \|f\|_{\mathfrak{L}^{\Phi_2}(0,1-2^{-j};X)} &\leq \|f\|_{\mathfrak{L}^{\Phi_2}(0,1;X)} \\ &\leq \|f\|_{L^{\Phi_2}(0,1;X)} = \inf_{\lambda>0} \frac{1}{\lambda} \int_0^1 \exp(\lambda^2 \|f(t)\|^2) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\|\xi_j\| &\leq \inf_{\lambda>0} \frac{1}{\lambda} \int_0^1 \mathbb{E} \exp(\lambda^2 2^j \|W(t+2^{-j}) - W(t)\|^2) dt \\ &= \inf_{\lambda>0} \frac{1}{\lambda} \mathbb{E} \exp(\lambda^2 \|W(1)\|^2). \end{aligned}$$

This may be estimated by expanding into a power series:

$$\begin{aligned} &\frac{1}{\lambda} \sum_{k \geq 0} \frac{\lambda^{2k}}{k!} \mathbb{E}\|W(1)\|^{2k} \\ &\leq \frac{1}{\lambda} \left[ 1 + \sum_{k \geq 1} \frac{\lambda^{2k}}{k!} (K\sqrt{2k} \mathbb{E}\|W(1)\|)^{2k} \right] \\ &\leq \frac{1}{\lambda} \left[ 1 + \sum_{k \geq 1} (2e[\lambda K \mathbb{E}\|W(1)\|]^2)^k \right], \end{aligned}$$

where  $K$  is an absolute constant from the Gaussian norm comparison result [9, Corollary 3.2], and we used  $k^k/k! \leq e^k$ . With the choice  $\lambda = (2eK\mathbb{E}\|W(1)\|)^{-1}$ , we find that  $\mathbb{E}\|\xi_j\| \lesssim \mathbb{E}\|W(1)\|$ .  $\square$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI, GUSTAF HÄLLSTRÖMIN  
KATU 2B, FI-00014 HELSINKI, FINLAND

*E-mail address:* `tuomas.hytonen@helsinki.fi`

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, SNIADÉCKICH 8, 00-950 WARSAW,  
POLAND

*E-mail address:* `m.veraar@impan.gov.pl`, `mark@profsonline.nl`